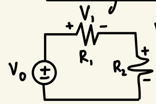
Charge and Current

$I = Q_v A v$, Q_v is charge density (C/m^3)

$$i = \frac{dq}{dt} = \frac{Q}{t}$$

$q = 1.6 \times 10^{-19}$ (sometimes e)

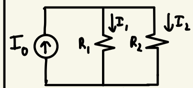
Voltage Divider



$$V_2 = V_{in}\left(\frac{R_2}{R_1+R_2}\right)$$

$$V_1 = V_{in}\left(\frac{R_1}{R_1+R_2}\right)$$

Current Divider



$$I_1 = I_0 \frac{R_2}{R_1+R_2}$$

$$I_2 = I_0 \frac{R_1}{R_1+R_2}$$

Concepts

Wheatstone Bridge can be used as a high precision ohmmeter

- WB consists of two voltage divider branches in parallel. Output is diff between these branches
- WB cancels common mode voltages, enhancing small voltage changes and making them detectable.

Voltage

$$V_{high} - V_{low} = \frac{W}{q}$$

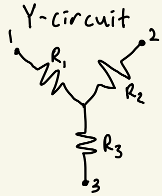
Power

$$P = \frac{dW}{dt} = \frac{dW}{dq} \cdot \frac{dq}{dt} = Vi$$

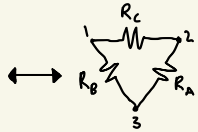
$$P = iV = i^2 R = \frac{V^2}{R}$$

Delta Y Transforms

Y-circuit



Δ Circuit



$Y \to \Delta$: $R_A = \dfrac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$

$$R_B = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

$$R_C = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

$\Delta \to Y$:

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \qquad R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$

$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C}$$

Waves

$v(t) = A\sin(kx \pm wt)$

- A = amplitude
- λ = wavelength
- k = wavevector $= \frac{2\pi}{\lambda}$
- T = period
- w = angular frequency $= \frac{2\pi}{T} = 2\pi f$
- f = frequency $= \frac{1}{T}$
- kx and wt are phases and are unitless

$v = f\lambda$

$$f = \frac{v}{\lambda} = \frac{v}{\frac{2L}{n}} = \frac{vn}{2L}$$

Standing waves: Two traveling waves in opposite directions

$f(0,t) = f(L,t) = 0.$

$A\sin(kx \pm wt)$

$v = A(\sin(kx)\cos(wt) \pm \cos(kx)\sin(wt))$

when $x=0, v=0$. First term becomes zero, second doesn't so

$v = A\sin(kx)\cos(wt)$

when $x = L$, $v=0$, so

$\sin(kL) = \sin(n\pi)$, where n is any integer

$kL = n\pi$

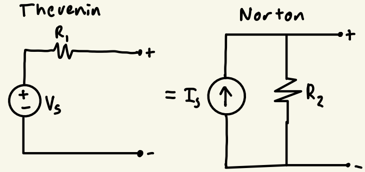
$\frac{2\pi}{\lambda} L = n\pi$

$\lambda = \frac{2}{n} L$

$\frac{\lambda}{2} = \frac{L}{n}$

Source Transformation

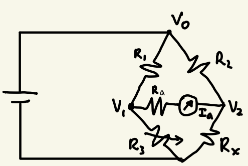
Thevenin $\qquad$ Norton



$R_1 = R_2$

$i_s = \frac{V_s}{R_1}$

Wheatstone Bridge Balanced Condition



Adjust R_3 until $I_a = 0$ to find R_x.

$V_1 = R_3 V_0 / (R_1 + R_3)$

$V_2 = R_x V_0 / (R_2 + R_x)$

$$\frac{R_3 V_0}{R_1 + R_3} = \frac{R_x V_0}{R_2 + R_x}$$

$$\frac{R_1 V_0}{R_1 + R_3} = \frac{R_2 V_0}{R_2 + R_x}$$

$$\frac{R_3}{R_1} = \frac{R_x}{R_2}$$

Balanced Condition: $R_x = \dfrac{R_2 R_3}{R_1}$

Mobility & Resistivity

Mobility: $\mu = \dfrac{q\,T_{avg}}{m}$ $\left(m^2/(V\cdot s)\right)$

Current Density: $J = nq\mu E$ (C/m^3)

n = free electron concentration (m^{-3})

E = electric field (V/m)

Conductivity: $\sigma = nq\mu$ (S/m)

Resistivity: $\rho = \dfrac{1}{nq\mu}$ (Ωm)

$m_e = 9.1 \times 10^{-31}$ kg $\qquad R = \rho \frac{L}{A}$

Mesh Analysis

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

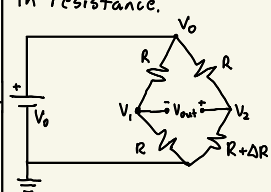
R_{kk} = sum all resistances in mesh k.

R_{jk} = negative of sum of shared resistors of meshes j and k.

V_k = sum of all voltage sources in k clockwise direction.

Wheatstone Bridge Sensor

$R_1 = R_2 = R_3$ for measuring small change in resistance.



$$V_1 = \frac{V_0 R}{R+R} = \frac{V_0}{2}$$

$$V_2 = \frac{V_0(R+\Delta R)}{R+R+\Delta R} = \frac{V_0(R+\Delta R)}{2R+\Delta R}$$

$$V_{out} = V_2 - V_1$$

$$= \frac{V_0(R+\Delta R)}{2R+\Delta R} - \frac{V_0}{2}$$

$$= \frac{2V_0(R+\Delta R) - V_0(2R+\Delta R)}{4R+2\Delta R}$$

$$= \frac{V_0 \Delta R}{4R\left(1+\frac{\Delta R}{2R}\right)}$$

$$\approx \frac{V_0}{4}\left(\frac{\Delta R}{R}\right)$$

KCL & KVL

Kirchhoff's current law: The algebraic sum of the currents entering a node must be zero.

Kirchhoff's voltage law: The algebraic sum of the voltages around a closed loop must always be zero.

Nodal Analysis

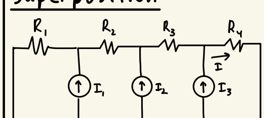
$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

G_{KK} = sum of all conductances $\left(\frac{1}{R}\right)$ connected to node k.

G_{jk} = negative of sum of all conductances $\left(\frac{1}{R}\right)$ between nodes j and k.

I_k = sum of current sources entering and leaving node k.

Superposition



$V_3 = I \cdot R_4$,

$$I = \frac{R_1}{R_1+R_2+R_3+R_4} I_1 + \frac{R_1+R_2}{R_1+R_2+R_3+R_4} I_2 + \frac{R_1+R_2+R_3}{R_1+R_2+R_3+R_4} I_3$$

Equivalent Resistance

Series: $R_{eq} = R_1 + R_2$

Parallel: $\dfrac{1}{R_{eq}} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

Capacitors and Capacitance

- Two electrodes separated by an insulator
- Initially, no voltage difference between electrodes
- When connected to a circuit, current flows through the circuit until electrodes have a voltage difference equal to the voltage source (KVL)
- $Q = CV$, where Q is accumulated charge and C is capacitance
- $C = \epsilon A/d$, ϵ is permittivity of material, A is area of the plate, and d is distance between the plates
- Permittivity of Vacuum, $\epsilon_0 = 8.854 \times 10^{-12}$ F/m

Magnitude of Transfer Functions

- Separate into parts, then take square root of all terms squared

Phase of Transfer Functions

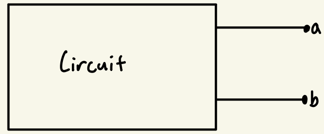
- Separate into parts, then take $\theta = \tan^{-1}\left(\frac{Imag}{Real}\right)$, add if multiple parts. Subtract num - denom for each part.

Thevenin Equivalent

To find V_{Th}, a and b should be unconnected. DO KCL!

To find I_{sc}, a and b should be connected only with a wire. DO KCL!

$$R_{Th} = \frac{V_{Th}}{I_{sc}}$$

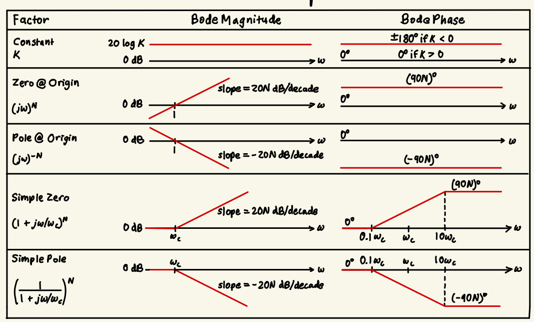


Inductors

- electrical components that store energy in a magnetic field when current flows

Bode plots

Find magnitude and phase equations then plug in ω = small, $\omega = \frac{1}{RC}$, ω = large and plot those points



Equivalent Capacitance

Series: $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$

Parallel: $C_{eq} = C_1 + C_2 + C_3$

Equivalent Inductance

Series: $L_{eq} = L_1 + L_2 + L_3$

Parallel: $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$

Current Flow in a Capacitor

$Q = CV$

$i = \frac{dq}{dt} = C \cdot \frac{dV}{dt}$

do KVL!

i = current flowing through the wires connected to capacitor

V = Voltage across capacitor

Equivalent Impedance

$Z_R = R$

$Z_C = \frac{1}{j\omega C}$

$Z_L = j\omega L$

Power Stored in Capacitors

$$P(t) = V \cdot i = V \cdot C \frac{dV}{dt}$$

$$\int_0^\infty P(t)\,dt = \int_0^{V_s} V \cdot C\,dV = \frac{1}{2} C V_2 \Big|_0^{V_s} = \frac{1}{2} C V_s^2$$

Time Constant

$\tau = R_{eq} C_{eq}$ or $\frac{L_{eq}}{R_{eq}}$

Voltage to dB

$dB = 20 \log_{10}(V)$

$V = 10^{dB/20}$

Phasor Notation

Convert everything to cosine

$\sin(x) = \cos(x - 90°)$

For each term:

$8\cos(6t - 45°) = 8e^{j-45°} = 8\angle -45°$

$8\angle -45° = 8(\cos(-45°) + j\sin(-45°))$

$= 5.66 - 5.66j$

Magnitude $= \sqrt{5.66^2 + (-5.66)^2}$

$= 8.00$

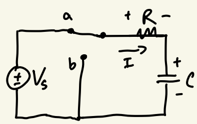
Phase Angle:

Draw real vs. imag graph

$\theta = \tan^{-1}(-5.66/5.66) = -45°$

Final phasor:

$8.00\angle -45° = 8e^{j-45°}$



charging @ $t = 0$

$V_s - IR - V_c = 0$

$V_s - RC\frac{dV_c}{dt} - V_c = 0$

$V_s - V_c = RC\frac{dV_c}{dt}$

$\frac{1}{V_s - V_c} dV_c = \frac{1}{RC} dt$

$\int_{V_c(0)}^{V_c(t)} \frac{1}{V_s - V_c} = \int_0^t \frac{1}{RC} dt$

$-\ln(V_s - V_c(t)) + \ln(V_s - V_c(0))$

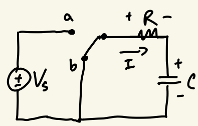
$= \ln\left(\frac{V_s - V_c(0)}{V_s - V_c(t)}\right) = \ln\left(\frac{V_s}{V_s - V_c(t)}\right)$

$= \frac{t}{RC}$

$\ln\left(\frac{V_s - V_c(t)}{V_s}\right) = -t/RC$

$1 - V_c(t)/V_s = e^{-t/RC}$

$V_c(t) = V_s(1 - e^{-t/RC})$



discharging @ $t = t'$

$-V_c - IR = 0$

$-V_c - RC\frac{dV_c}{dt} = 0$

$-V_c = RC\frac{dV_c}{dt}$

$\frac{1}{-V_c} dV_c = \frac{1}{RC} dt$

$\int_{V_c(t')}^{V_c(t)} \frac{1}{-V_c} dV_c = \int_{t'}^{t} \frac{1}{RC} dt$

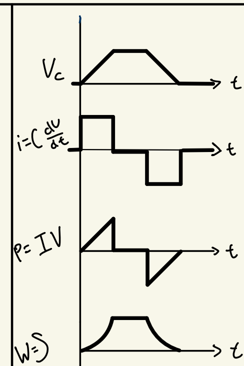
$-\ln V_c(t) + \ln V_c(t')$

$= \ln\left(\frac{V_c(t')}{V_c(t)}\right) = \frac{t - t'}{RC}$

$\frac{V_c(t')}{V_c(t)} = e^{(t - t')/RC}$

$V_c(t') = V_c(t)\, e^{(t-t')/RC}$

$V_c(t) = V_c(t')\, e^{-(t-t')/RC}$



$V_L = L\, di/dt$

$\int_0^t \frac{V_L}{L} dt = \int_{i(0)}^{i(t)} 1\, di$

$\frac{1}{L}\int_0^t V_L\, dt = i(t) - i(0)$

$i(t) = i(0) + \frac{1}{L}\int_0^t V_L\, dt$

$L = \frac{\mu N^2 S}{\ell}$

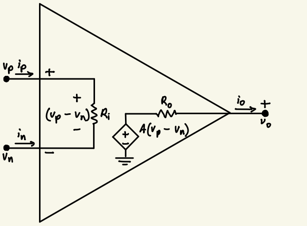
$\mu_r = \frac{\mu}{\mu_0}$

N = # coils

S = cross-sectional area

$\mu_0 = 4\pi \cdot 10^{-7}$ H/m

Ideal Op Amp Conditions



$V_o = A(V_p - V_n)$ b/c $R_o = 0$

$(V_p - V_n) = \frac{V_o}{A}$

if $A \to \infty$, $V_p - V_n = 0$

① $V_p = V_n$

$(V_p - V_n) = i_p R_i$

if $R_i \to \infty$

② $i_p = 0$

Negative Feedback

Output terminal is connected to the V_n terminal

Initial and Final Conditions for Capacitors and Inductors

① At steady state, all voltages and currents in the circuit reach some fixed value.

$\frac{dV}{dt} = 0$, $\frac{di}{dt} = 0$

② The voltage through a capacitor and current through an inductor cannot change instantaneously.

-requires infinite i_c or V_L

	Initial Conditions ($t = 0^+$)	Final Conditions ($t = \infty$)
Capacitors	$V_c(t = 0^+) = V_c(t = 0^-)$	$i_c(\infty) = C\frac{dV_c}{dt} = 0 \to$ open
Inductors	$i_L(t = 0^+) = i_L(t = 0^-)$	$V_L(\infty) = L\frac{di_L}{dt} = 0 \to$ short

Second Order Diff EQs

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

$x(t)$ = quantity of interest

α = damping factor

ω_0 = undamped natural frequency

Characteristic Polynomial:

$s^2 + 2\alpha s + \omega_0^2 = 0$

Homogeneous solution:

$x_h(t) = Ae^{s_1 t} + Be^{s_2 t}$

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ $\quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

Cases:

① Overdamped: $\alpha > \omega_0$

$x_h(t) = Ae^{s_1 t} + Be^{s_2 t}$

② Underdamped: $\alpha < \omega_0$

$x_h(t) = e^{-\alpha t}(C\cos(\omega_d t) + D\sin(\omega_d t))$

$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

③ Critically damped: $\alpha = \omega_0$

$x_h(t) = Ae^{-\alpha t} + Bte^{-\alpha t}$

$x(t) = x_h(t) + x_p(t)$

$x_p(t) = x(t = \infty)$, analyze the circuit in steady state.

Use initial conditions $x(0)$ and $x'(0)$ to solve for A & B / C & D

Current Flow in an Inductor

$V_L = L\frac{di_L}{dt}$ do KCL!

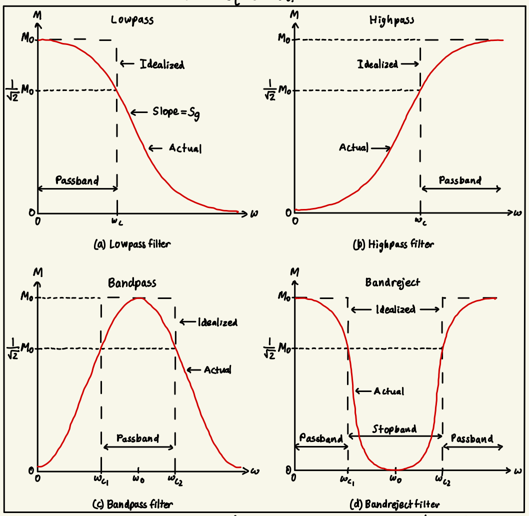
Filters

① Lowpass filters are idealized for low frequencies

② Highpass filters are idealized for high frequencies

③ Bandpass filters are idealized for a range of frequencies centered at ω_0, but cuts off high and low frequencies

④ Bandreject filters are idealized for high and low frequencies but not intermediate frequencies.



(a) Lowpass filter (b) Highpass filter (c) Bandpass filter (d) Bandreject filter

Resonant Frequency, ω_0: frequency at which $|H(\omega)|$ reaches its maximum

Bandwidth, B: frequency range extending between ω_{c1} and ω_{c2}, where ω_{c1} and ω_{c2} are where $|H(\omega)| = \frac{max}{\sqrt{2}}$ or about -3dB below the max

R/L

Quality factor, Q: measures how sharp or selective the resonance is.

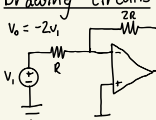
$Q = \frac{\omega_0}{B} = \frac{\omega_0 L}{R}$

Higher $Q \to$ narrower bandwidth $\to$ higher selectivity

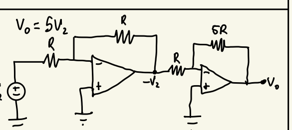
Lower $Q \to$ wider bandwidth $\to$ lower selectivity

Drawing Circuits

$V_o = -2V_1$

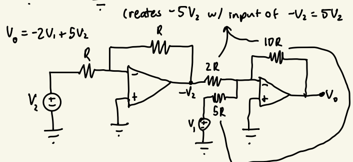


$V_o = 5V_2$



(creates $-5V_2$ w/ input of $-V_2 = 5V_2$)

$V_o = -2V_1 + 5V_2$



creates $-2V_1$

RC Circuits

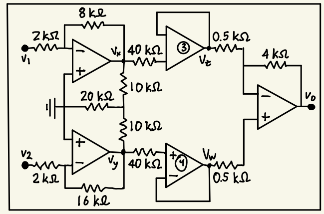
Charging: $V_c(t) = V_0(1 - e^{-\frac{t}{\tau}})$

Discharging: Calculate constant in terms of e, then combine w/ other e term. Factor exponent to get time shift.

Steady State

After current has been flowing through a capacitor for a long time, the capacitor will be fully charged and act as an open circuit. $\to i_c = 0, V_c = V_0$

After current has been flowing through an inductor for a long time, the inductor will act as a short circuit. $\to V_L = 0, i_L = I_0$



$V_x = -\frac{8\,k\Omega}{2\,k\Omega}V_1 = -4V_1$

$V_y = -\frac{16\,k\Omega}{2\,k\Omega}V_2 = -8V_2$

③ and ④ are voltage followers since their feedback loop has no resistance.

So $V_z = V_x = -4V_1$

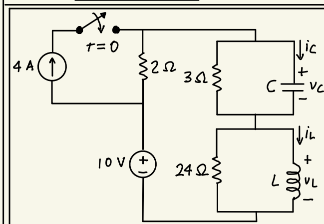
$V_w = V_y = -8V_2$

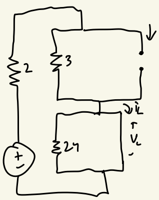
⑤

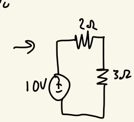
Negative Terminal: $V_o = -\frac{4\,k\Omega}{0.5\,k\Omega} \cdot -4V_1 = 32V_1$

Positive Terminal: $V_o = \frac{(4+0.5)\,k\Omega}{0.5\,k\Omega} \cdot -8V_2 = -72V_2$

$\boxed{V_o = 32V_1 - 72V_2}$



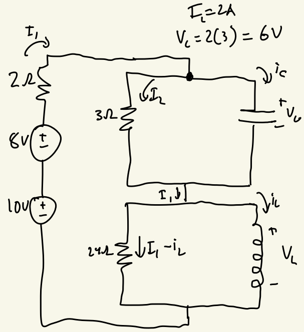




$10 - 5I = 0$

$I = 2A$

$V_c = 2(3) = 6V$



$18 - 2I_1 - V_c - 24(I_1 - I_L) = 0$

$18 - 2I_1 - 6 - 24(I_1 - 2) = 0$

$I_1 = \frac{30}{13}$ A

$\frac{30}{13} = i_c + \frac{V_c}{3} = i_c + 2$

$i_c = \frac{4}{13}$ A $\to V_L = 24 \cdot \frac{4}{13} = \frac{96}{13}$ V

since all current will flow through the capacitor

Design a series RLC bandpass filter with a center frequency $f_0 = \text{MHz}$ and a quality factor $Q=20$, given that $L = 0.1\,\text{mH}$

$$\omega_0 = 2\pi f_0 = 2\pi \times 10^6$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-4}C}} = 2\pi\times10^6 \rightarrow C = 0.25\,\text{nF}$$

$$20 = \frac{2\pi\times10^6 \cdot 10^{-4}}{R} \rightarrow R = 31.4\,\Omega$$

Determine the 10 dB bandwidth of the filter, which is defined as the bandwidth between frequencies at which the power level is 10 dB below the peak value.

$$M_{BP}[dB] = -10\text{ dB}$$

$$20\log_{10} M_{BP} = -10\text{ dB}$$

$$M_{BP} = 10^{-0.5} = 0.316$$

$$M_{BP} = \frac{\omega RC}{\sqrt{(1-\omega^2LC)^2 + \omega^2R^2C^2}}$$

$$\omega_{ab} = 5.87\times10^6,\ 6.81\times10^6$$

$$\boxed{B_{10\,dB} = 9.4\times10^5\ \text{rad/s}}$$

In Hz: $\omega_a = 5.87\times10^6$, $\omega_b = 6.81\times10^6$

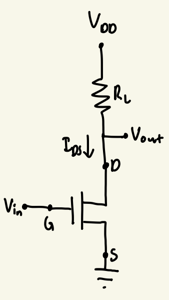
$$\frac{\omega_a}{\omega_0} = 0.93,\quad \frac{\omega_b}{\omega_0} = 1.08$$

$$\omega_a - \omega_b = 2\pi(f_a - f_b)$$

$$\omega_0 = 2\pi f_0 \rightarrow \frac{\omega_a - \omega_b}{\omega_0} = \frac{2\pi(f_a - f_b)}{2\pi f_0}$$

$$\frac{\omega_a - \omega_b}{\omega_0}\cdot f_0 = (f_a - f_b)$$

$$\boxed{B_{10\,dB} = 0.15\,\text{MHz}}$$



$R_L = 0.1\,\text{k}\Omega$

when $V_{GS} = 1V$, $I_{DS} = 1\,\text{mA}$

when $V_{GS} = 0V$, $I_{DS} = 1\times10^{-9}\,\text{A}$

$$V_{out} = V_{DD} - I_{DS}R_L$$